IASS-IACM 2000

Fourth International Colloquium on
Computation of Shell & Spatial Structures

June 5-7, 2000
Chania-Crete, Greece

A HISTORY OF THE PRINCIPAL DEVELOPMENTS AND APPLICATIONS OF THE FORCE DENSITY METHOD IN GERMANY 1970-1999

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Key words: Force-density, Form-finding, Analysis, Textile, Tensile & Funicular Structures

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Abstract. *This paper describes the historical development of Force Density Method techniques for analytical form-finding, statical analysis and the determination of workshop drawings of lightweight structures from 1970 to the present day. Particular attention is directed to the key developments made at The University of Stuttgart's Institute for the Application of Geodesy in Civil Engineering (IAGB); at the Technical University of Berlin's Institute of Geodesy and Geomatics; and within the specialised software company technet GmbH. Attention is concentrated on the calculation of mechanically prestressed cable nets and membranes, minimal surfaces and pneumatically prestressed constructions.*

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1 Introduction

1.1 The Situation Before 1970

Although prestressed structures like tents belong to the first man-made structures, little was known about the analytical modelling of their load deflection behaviour. Prestressed cable-nets and textile membranes are characterised by the inherent interaction between their geometry and stress distribution. This relationship between the form and forces makes it impossible to directly design such structures as is the case with conventional structures.

When in 1967 the German Pavillion for the Montreal EXPO was built, no practical analytical solution technique was available to determine the cable-net form, cutting pattern and behaviour under external load. At that time the only way for the design and the realisation of such nets was to use physical models.

Finding a feasible form requires the determination of a figure of equilibrium of inner forces and loads for the structure. This typically results in a doubly curved surface. Mathematically these surfaces could only be roughly approximated by differential equations. As doubly curved surfaces can not be flattened without distortion, the generation of precise cutting patterns is required for fabrication. Finally the structure will undergo large deflections under acting loading conditions which means that analytical methods would have to be able to cope with that.

1.2 Analytical Form-finding and the Force Density Method

To design the cable-net roofs for the 1972 Munich Olympic Games stadium, Frei Otto built precise physical models which were intended to be the source of information for all relevant data. Linkwitz proposed to measure the models precisely applying close range photo-grammetric methods which would allow for a simultaneous determination of the 3D-geometry of the model without touching it. It was realised however that the models were by no means precise enough to derive the cutting pattern for an equal mesh cable net made from steel.

The photogrammetric measurements of the physical models had to be modified in order to fulfil the constraints of equal unstressed mesh-width and of force equilibrium at each node. The analytical solution for this task was achieved by applying the method of least squares to the measured nodal coordinates [1], [4] observing the boundary conditions above. Applying this technique, the cutting pattern for the stadium roof was created in a time consuming but successful procedure using all the computer power available at that time.

In 1971 Linkwitz and Schek [1] discovered a new formulation of the figure of equilibrium of forces, the force-density formulation. They realised that this was more appropriate for solving the problem, especially that of finding good initial geometry.

Figure 1: Munich Olympic Stadia cable-net model.

In order to solve the numerous open problems associated with wide-span prestressed structures, an interdisciplinary research group at Stuttgart University was established and funded by the German Research Foundation. This was the SFB 64 on Lightweight Structures.

2 Force Density Algorithms for Pin-jointed Networks

2.1 Linear Form-finding

The properties of the Force Density Method were subsequently studied thoroughly [2], [4] and the method could be implemented in an efficient way by applying special sparse matrix techniques for solving the resulting equations. It proved to be a powerful tool for setting up and solving the equations of equilibrium for prestressed networks and structural membranes, without requiring any initial coordinates of the structures [3], [5].

The essential ideas are as follows. Pin-jointed network structures assume the state of equilibrium when internal forces *s* and external forces *p* are balanced

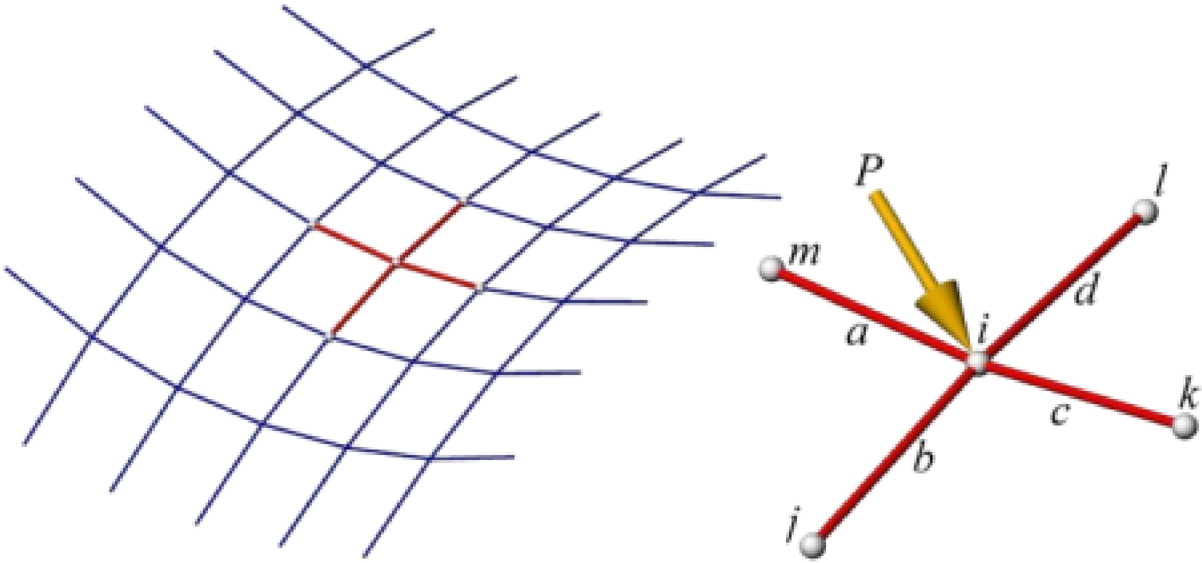

Figure 2. Part of a cable network.

In the case of node *i* in Figure 2, the equations read,

$$\begin{aligned} s_a \cos(a,x) + s_b \cos(b,x) + s_c \cos(c,x) + s_d \cos(d,x) &= p_x \\ s_a \cos(a,y) + s_b \cos(b,y) + s_c \cos(c,y) + s_d \cos(d,y) &= p_y \\ s_a \cos(a,z) + s_b \cos(b,z) + s_c \cos(c,z) + s_d \cos(d,z) &= p_z \end{aligned}$$

where s_a, s_b, s_c and s_d are the bar forces and cos(*a*,*x*) are the projection lengths of the normalised cable lengths on the *x*-axis. Substituting the above cosine values by the normalised projection lengths of the form $(x_m - x_i)/a$, results in,

$$\begin{aligned} \frac{s_a}{a}(x_m - x_i) + \frac{s_b}{b}(x_j - x_i) + \frac{s_c}{c}(x_k - x_i) + \frac{s_d}{d}(x_l - x_i) &= p_x \\ \frac{s_a}{a}(y_m - y_i) + \frac{s_b}{b}(y_j - y_i) + \frac{s_c}{c}(y_k - y_i) + \frac{s_d}{d}(y_l - y_i) &= p_y \\ \frac{s_a}{a}(z_m - z_i) + \frac{s_b}{b}(z_j - z_i) + \frac{s_c}{c}(z_k - z_i) + \frac{s_d}{d}(z_l - z_i) &= p_z \end{aligned}$$

In these equations, the lengths *a*, *b*, *c* and *d* are non-linear functions of the coordinates. In addition, the forces are dependent on the unstressed mesh widths and on Hooke's law. Substituting these functions would lead to a finite element formulation as shown in [4].

Based on the interaction of form and forces, the form-finding process aims to receive the geometry of a form with a desired prestress and a surface discretisation of a desired mesh-width. This initial information can be brought together in force density parameters $s_a / a = q_a$, for every link. The resulting linear equations read,

$$
\begin{aligned}
q_a(x_m - x_i) + q_b(x_j - x_i) + q_c(x_k - x_i) + q_d(x_l - x_i) &= p_x \\
q_a(y_m - y_i) + q_b(y_j - y_i) + q_c(y_k - y_i) + q_d(y_l - y_i) &= p_y \\
q_a(z_m - z_i) + q_b(z_j - z_i) + q_c(z_k - z_i) + q_d(z_l - z_i) &= p_z
\end{aligned}
$$

The system of equations assembled is extremely sparse and can be efficiently solved for the coordinates of the structure using the *Conjugate Gradient* method as described in [4].

2.2 Geometrically Non-linear Statical Analysis

As shown in [2] and [4], the *Force Density* method could be extended efficiently, in order to efficiently perform the computation of load cases which leads to a non-linear analysis. The set of formulae of the *Finite Element Method* can be derived directly from the *Force Density*'s formulation. However, the *Force Density* method has been proven to be numerically more stable for the calculation of lightweight structures where large deflections often occur, and where parts of the structure become slack [6].

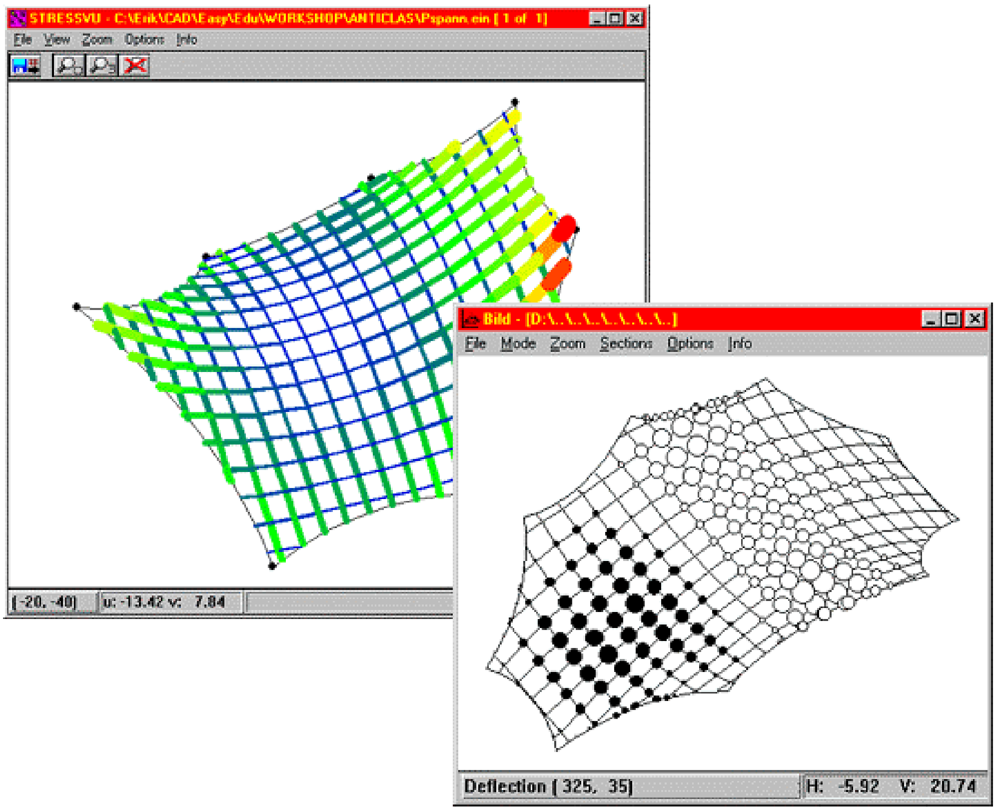

Figure 2. Stress and deflection visualisation after geometrically non-linear load analysis [17].

Prior to any statical analysis, the structure form-found by the linear *Force Density* equations, has to be materialised. This can be done without disturbing the state of equilibrium of forces. For any set of positive q-values, fixed positions and given external forces p there exists a unique set of coordinates of the nodes of the equilibrated structure. Applying Hooke's law for the bar force s_a results in,

$$s_a = EA\frac{a - a_0}{a_0}$$

where A is the cross sectional area of bar a, E the modulus of elasticity, and a_0 is the unstressed length of bar a. Substituting s_a by q_a according to $q_a = s_a / a$ results in,

$$a_0 = \frac{EAa}{q_a a + EA}$$

As a is a function of the coordinates of the bar ends, the materialised unstressed length is a function of q_a and the coordinates. For each q_a there exists a corresponding unstressed bar length a_0.

In order to perform statical structural analysis subject to various external loads, the unstressed lengths have to be kept fixed. This can be achieved mathematically by extending the *Force Density* system of equations of equilibrium by the materialisation equations shown above. The enlarged system of equations is no longer linear in coordinates and force densities. It has to be linearised using initial values for these parameters. As the *Force Density* information from the linear form-finding procedure is naturally available, this proves to be extremely favourable for achieving convergence. The use of *Force Densities* as pre-information in the non-linear problems of statical analysis results in powerful convergence characteristics.

2.3 T-elements

For an application of the method of force densities to form-finding, the main problem was not any more of how to get initial coordinates of a structure but how to discretise the surface in the best way. An automated discretisation process of the surface often resulted in nice regular meshes inside the structure but unrealistically short connections of the regular network to the boundary. These short bar connections often built up kinks during the analysis. Correcting the net interconnection was a time-intensive part in the form-finding process.

In reality a structural membrane has a continuous surface and there is no need to have common nodes on the edge cables. When two sections of a network representing a membrane had a common edge cable, the corresponding computer-model developed zig-zag line. Enforcing the edge-points of corresponding cables of the individual sections to one and the same node would be time consuming and not solve the problem adequately.

In [9] the proposal was made to separate the edge cable discretisation from the structure of the regular inner net. The connection points of the inner net to the boundary would be split proportional to their distance to the neighbouring discretisation points on the edge cable. This is achieved simply by replacing the end position of each net connector by the adjacent common boundary discretisation nodes. In Fig.2 node *i* is replaced by *j* and *k* and prefixed proportional values *m* and *p* which force point *i* to be between *j* and *k*.

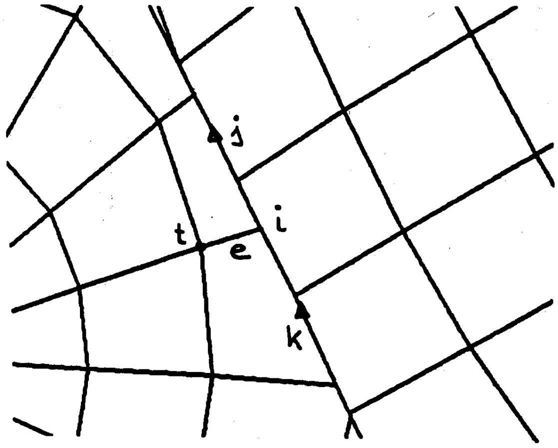

Figure 3. Nodes of discretisation and end nodes of cables

This reads:

$$\begin{aligned} x_t &= mx_j + px_k \\ y_t &= my_j + py_k \\ z_t &= mz_j + pz_k \end{aligned}$$

The concept has proven to be successful in the design of sensitive tensile structures [10] and even works for very short bar connections.

3 Membrane Models and Minimal Surfaces

Two discretisation strategies are mainly taken: *Equivalent Cable-net Link* and *Triangular*.

3.1 Cable-net Link Discretisation

The task of form-finding of textile membranes may be solved by form-finding a prestressed net structure. The continuous membrane is approximated by a discretised grid of cables, The force densities are derived from the desired prestress and the area of influence as described earlier in this paper.

This model is valid if the fabric consists of woven material which typically has negligable shear stiffness. The directions of weft and warp should coincide with the directions of principal surface curvature. For load analysis the stiffnesses of the discretised bar-elements can be derived from the area of influence of the discretised surface.

In order to model the Poisson interaction of material properties between warp and weft cruciform elements can be applied [12]. Typically this is counter-productive as it results in an increased model complexity without increased accuracy in practice due to the inability to accurately calibrate the materials.

3.2 Triangular Discretisation

Where the shear-stiffness of the material is significant, triangular elements can be applied. It was shown in [13] that the method of force densities is also valuable in these formulations.The mathematical model described in [13] is based on a geometrical and physical description of the undeformed and deformed state of all triangles of the network. The triangles will undergo deformations caused by the forces acting in the structure.

The nodal coordinates of the state of equilibrium are defined by the minimum of the inner energy (the summarised energies of all the triangles) and the negative potential of the external forces. As a result, the sum of internal and external forces affecting the nodes, tend to zero when approaching the figure of equilibrium.

3.3 Membrane Elastic Analysis

For general membranes the inner energy to minimised can be read as:

$$\Pi = \frac{1}{2}\int_F \sigma^t \varepsilon dF \rightarrow \min .$$

The material description according to St. Venant-Kirchhoff models a linear relation between the 2nd Piola-Kirchhoff stress vector and Green's strain vector. It therefore follows that,

$$\begin{bmatrix} \sigma_{uu} \\ \sigma_{vv} \\ \sqrt{2}\sigma_{uv} \end{bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & 0 \\ & E_{2222} & 0 \\ & & 2E_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{uu} \\ \varepsilon_{vv} \\ \sqrt{2}\varepsilon_{uv} \end{bmatrix}$$

where four parameters are defined: the modulus of elasticity in the *u* and *v* directions E_{1111}, and E_{2222}, the modulus of elasticity E_{1122}, which models transverse loading and the shear modulus $2E_{1212}$.

3.4 Minimal Surface Structures

Minimal surfaces are also characterised by the property of a minimal area within a three-dimensional given boundary. Experimentally such surfaces can be produced by dipping a free shaped frame into a soap liquid. Lifting the frame, a soap film might fill the frame. Due to the small influence of the gravity, the soap film represents a minimal surface with a zero mean curvature and equal surface stress.

Minimal surfaces can be generated numerically based on a triangular network description of nodes and links. Given a polygonal boundary curve and the inner area represented by triangles, the search for a minimal surface can be achieved by minimising the sum of the areas of all triangles [8]. This efficient and robust implementation deals with flexible boundary configurations as well as fixed curves.

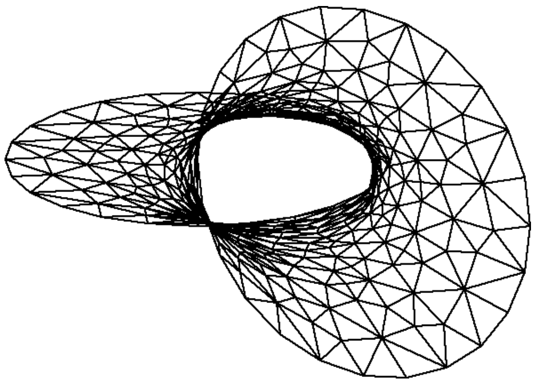

Figure 4. Discretised minimal surface.

Introducing a fictitious stress variable σ, the minimum formulation can be defined as a state of equilibrium [13]. The inner energy Π of the surface is given by,

$$\Pi = \sigma \sum_{i=1}^{n} F_{\Delta i} \rightarrow \min .$$

where F is the area of one of the n triangles. As shown in [8], it is favourable to use Heron's quadratic representation for modelling the area of the triangles. In terms of the triangle edge lengths l_1, l_2 and l_3, the area reads,

$$16F_\Delta^2 = -l_1^4 - l_2^4 - l_3^4 + 2l_1^2 l_2^2 + 2l_2^2 l_3^2 + 2l_3^2 l_1^2 = \begin{bmatrix} l_1^2 & l_2^2 & l_3^2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ & -1 & 1 \\ & & -1 \end{bmatrix} \begin{bmatrix} l_1^2 \\ l_2^2 \\ l_3^2 \end{bmatrix}$$

The minimisation of the energy leads to the equations of equilibrium for the nodal coordinates of the net. For the calculation of minimal surfaces the undeformed triangle geometries are not required.

According to [13] minimal surfaces can be generated applying the following membrane material properties,

$$\begin{bmatrix} E_{1111} & E_{1122} & 0 \\ & E_{2222} & 0 \\ & & 2E_{1212} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & -1 \end{bmatrix}$$

As the undeformed geometry of the triangles do not enter the equations, this membrane approach can be regarded as analogous to the force-density formulation for prestressed nets, where the undeformed bar element lengths, are not required in the form-finding process.

3.5 Volume Constrained Pneumatic Structures

Bubbles can be seen as minimal surface structures enclosing a given volume. By enforcing a constant volume a specific inner pressure will result for each resulting chambers. The volume of the chamber may be modelled by tetrahedrons, built by the surface triangles and the centre of gravity of each chamber. The sum of all tetrahedrons gives the volume of one chamber

$$V = \sum_{i=1}^{n} V_{\Delta i}$$

The energy of the bubble with a given volume V_c can be formulated by:

$$\Pi = \sigma \sum_{i=1}^{n} F_{\Delta i} - k(V - V\mathrm{c}) \rightarrow \min .$$

In [13] it was shown that the Lagrangian multiplier k is identical to the inner pressure of the chamber.

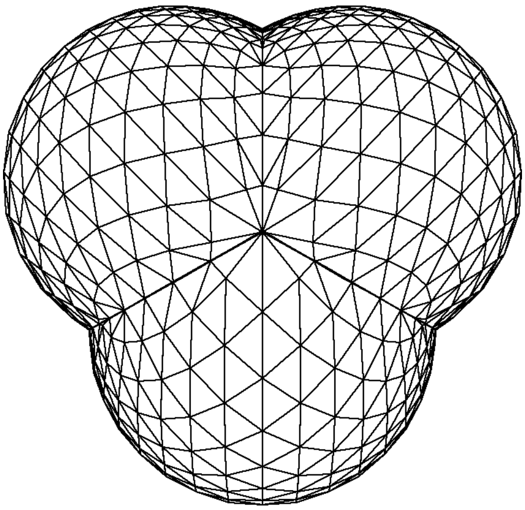

Figure 5. Chambered pneumatic structure.

4 Design of Funicular Grid-shells

4.1 Form-finding

The form-finding of tensile networks subject to vertical self-weight loading results in forms which will experience pure compression when subjected to inverted self-weight loading. Such compressive funicular structures were widely used by Gaudi.

Figure 5. Bad Dürrheim funicular timber grid shell under construction.

4.2 Geometrically Non-linear Frame Analysis

Subsequent to tensile funicular form-finding under inverted self weight loading it is necessary to perform frame stress analysis of the design subject to applied loading. In the case of heavy rigid structures this can be performed using conventional analysis software. Where the design is composed of light flexible members, as is typical with timber structures, geometrically non-linear systems must be used. One of the biggest problems with applying standard FEM software to this problem is the issue of link slackening on-off non-linearity. By using *Force Density* techniques a very robust and integrated solution strategy for this problem was developed in [14].

5 Current Developments

Today attention is being focused on dealing with the large variety of complicated hybrid structures which are technologically feasible. For example dealing with structures combining pre-stressed membranes with pneumatic or flexible spline stiffeners is very complex. Similarly in some structural projects it is necessary to prescribe sophisticated constraints beyond the simple axial planes. For example, the problem of a membrane sliding over a grid-shell requires that the membrane to shell on-off contacting problem be solved, as well as inter-surface frictional modelling.

References

[1] Linkwitz, K. and Schek, H.-J., (1971), \`Einige Bemerkungen zur Berechnung von vorgespannten Seilnetzkonstruktionen,´ *Ingenieur-Archiv* 40, 145-158.

[2] Schek, H.-J., (1974), \`The force density method for form finding and computation of general networks,´ *Computer Methods in Applied Mechanics and Engineering* 3, 115-134.

[3] Gründig, L., Schek, H.-J., (1974), \`Analytical Form Finding and Analysis of Prestressed Cable Networks,´ International Conference on Tension Roof Structures, London, April 1974.

[4] Gründig, L., (1975), \`Die Berechnung von vorgespannten Seilnetzen und Hängenetzen unter Berücksichtigung ihrer topologischen und physikalischen Eigenschaften und der Ausgleichungsrechnung,´ DGK Reihe C, Nr. 216, 1976 and SFB 64-Mitteilungen 34, 1976.

[5] Gründig, L., Hangleiter, U., (1975), \`Computation of prestressed cable-nets with the force densities method,´ IASS-Symposium *Cable Structures*, Bratislava.

[6] Gründig, L., (1985), The FORCE-DENSITY - Approach and Numerical Methods for the Calculation of Networks. Proc. of 3. Intern. Symposium *Weitgespannte Flächentragwerke*, Stuttgart, March 1985.

[7] Gründig, L., Bahndorf, J., (1986), `Formfinding of a Roof Structure for a Health Spa,´ First International Conference on Lightweight Structures in Architecture, Sydney, Australia, August 1986.

[8] Gründig, L., (1988), `Minimal Surfaces for Finding Forms of Structural Membranes". *Civil-Comp 87*, London, Vol.2, pp 109-114, and *Computers and Structures* **3** 1988.

[9] Gründig, L. and Bahndorf, J., (1988), `The Design of Wide-Span Roof Structures Using Micro-Computers,´ *Computers & Structures* 30, 495-501.

[10] Linkwitz, K., Gründig, L., Bahndorf, J., Neureither, M. and Ströbel, D., (1988), `Optimizing the Shape of the Roof of the Olympic Stadium, Montreal,´ *Structural Engineering Review* 1, 225-232.

[11] Gründig, L. and Moncrieff, E., (1993), `Formfinding of Textile Structures,´ in *Proc. Studiedag-Seminaire Textielstrukturen Architecture Textile*, Vrije Universiteit Brussel, 25th May 1993.

[12] Bauerle, J., (1995), *Ein Beitrag zur Berechnung des Zuschnitts von vorgespannten Membranen*, DGK, Reihe C, Nr. 439.

[13] Singer, P., (1995), *Die Berechnung von Minimalfläachen, Seifenblasen, Membrane und Pneus aus geodätischer Sicht*, DGK, Reihe C, Nr. 448.

[14] Ströbel, D., (1997), *Die Anwendung der Ausgleichungsrechnung auf elastomechanische Systeme*, DGK, Reihe C, Nr. 478.

[15] Gründig, L. and Moncrieff, E., (1998), `Formfinding, Analysis and Patterning of Regular and Irregular-Mesh Cablenet Structures,´ in Hough, R. and Melchers, R. (Eds.), *LSA98: Lightweight Structures in Architecture Engineering and Construction* Proceedings IASS 39th Congress, October, 1998, Sydney, Australia, IASS/LSAA.

[16] Moncrieff, E., Gründig, L. and Ströbel, D., (1999), `The Cutting Pattern Generation of the Pilgrim's Tents for Phase II of the Mina Valley Project,´ in Astudillo, R. and Madrid, A. J. (Eds.), *Proc. IASS 40th Anniversary Congress*, September 20-24, 1999, Madrid, Spain, IASS/CEDEX.

[17] technet GmbH, (2000), *Easy, User manual for integrated surface structure design software*, technet GmbH, Maassenstr. 14, D-10777 Berlin, Germany, http://www.technet-gmbh.com